

# New static spheroidal solution in Jordan-Brands-Dicke theory.

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## Abstract

The static spheroidal solutions of Jordan-Brands-Dicke theory (JBD) are studied. We consider the effect of the anisotropic stresses of scalar field on the shape of JBD self-gravitating objects. It is shown that scalar fields can have significant effect on the structure and properties of self-gravitating objects. In contrast with general relativity in JBD theory there are nonflat static spheroidal solutions.

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## 1 Introduction

A wide spread assumption in the study of stellar structure is that the shape of star can be modeled as a spherical symmetry object. This approach has been used extensively in the study of star, star system and galaxies [1]. However, in many systems, deviation from spherical symmetry may play an important role in determining of them properties. Physical situation where unspherical shape may be relevant are very diverse. Scalar self-gravitating objects resulting from the non minimal coupling scalar fields to gravity are a system where anisotropic pressure occurs naturally [2]. A model for the Universe where the dark matter and energy are the scalar nature can be realistic and could explain most of the observed structures [3]. The self-interaction of the scalar field could explain the behavior of galaxy rotation curves all along the background.

Anisotropy appears as an extra assumption on the behavior of scalar fields and on the shape of equilibrium configuration. Since we still do not have a formulation of the possible anisotropic stresses is emerging in these or other contexts, we take the approach of finding several exact solutions representing physical situations, modelled by ellipsoid of revolution. Our goals hear is to find exact spheroidal solution, offering an analysis of the change in the physical properties of the stellar and galaxy models due to presence of non minimally coupled scalar fields. In this context, particularly interesting is the case of JBD theory [4], [5] where pressure anisotropies come in action.

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## 2 Static spheroidal vacuum solutions of Jordan-Brans-Dicke theory.

JBD theory can be thought of as a minimal extension of general relativity designed to properly accommodate both Mach's principle and Dirac's large number hypothesis. The progress in the understanding of scalar-tensor theories of gravity is closely connected with finding and investigation of exact solutions. Shortly after JBD theory was proposed, Heckmann obtained parametric form of the exact static vacuum solution to the JBD equations [7]. Later Brans [5] find the static, spherically symmetric, vacuum solution of the JBD equations in isotropic coordinates. In the Jordan conformal frame, the JBD action takes the form [4] (we use geometrized units such that  $G = c = 1$  and we follow the signature  $+, -, -, -$ ).

$$S = \int dx \sqrt{-g} (\phi R - \omega g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi) + S_m. \quad (1)$$

Here,  $\omega$  coupling constant,  $R$  is the Ricci scalar curvature with respect to the space-time metric  $g_{\mu\nu}$  and  $S_m$  denotes action of matter fields. (We use units in which gravitational constant  $G=1$  and speed of light  $c=1$ .)

Variation of (1) with respect to  $g_{\mu\nu}$  and  $\phi$  gives, respectively, the field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{1}{\phi} T_{\mu\nu}^M + T_{\mu\nu}^{JBD}, \quad (2)$$

where

$$\begin{aligned} T_{\mu\nu}^{JBD} = & \left[ \frac{\omega}{\phi^2} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla^\alpha \phi \right) + \right. \\ & \left. + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \phi) \right], \end{aligned} \quad (3)$$

and

$$\nabla_\alpha \nabla^\alpha \phi = \frac{T_\lambda^{M\lambda}}{3 + 2\omega}, \quad (4)$$

and  $T_\lambda^{M\lambda}$  is the energy momentum tensor of ordinary matter which obeys the conservation equation  $T_{\mu\nu;\lambda}^{M\lambda} g^{\nu\lambda} = 0$ . One can derive the energy density and pressure of JBD field from (3)  $\rho^{JBD} = T_{00}$ ,  $P^{JBD} = T_{ii}$  ( $i = 1, 2, 3$ ).

As we have already mentioned we consider standard static spheroidal space-time. Solutions to the equation in spheroidal coordinates have application to a wide range of problems in physics [6]. The two-dimensional elliptic coordinate system is defined from the set of all ellipses and all hyperbolas with a common set of two focal points. Oblate spheroidal coordinates are derived from elliptic coordinates by rotating the elliptical coordinate system about the perpendicular bisector of the focal points. Similarly,

one can obtain the prolate spheroidal coordinates by rotating it about the parallel bisector. We adopt coordinates that allow us to write spheroidal geometry in prolate form

$$ds^2 = -B(\xi) dt^2 + A(\xi) \left( c \sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}} d\xi^2 + c \sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}} d\eta^2 + c \sqrt{(\xi^2 - 1)(1 - \eta^2)} d\varphi^2 \right), \quad (5)$$

and in oblate form

$$ds^2 = -B(\xi) dt^2 + A(\xi) \left( c \sqrt{\frac{\xi^2 - \eta^2}{\xi^2 - 1}} d\xi^2 + c \sqrt{\frac{\xi^2 - \eta^2}{1 - \eta^2}} d\eta^2 + c \xi \eta d\varphi^2 \right), \quad (6)$$

where A, B are function of  $\xi$  and  $\xi \geq 1$ ,  $-1 \leq \eta \leq 1$ ,  $0 \leq \varphi \leq 2\pi$ . We denote the separation of the two focal points by  $c$ .

Then the solutions of the gravitational field equations in the vacuum  $T_{\mu\nu}^M = 0$  take in oblate case the form

$$A = a_0 \left( \frac{1 + \sqrt{1 - \xi^2}}{-1 + \sqrt{1 - \xi^2}} \right)^{\beta \left( 1 + \frac{1}{\sqrt{-3-2\omega}} \right)}, \quad (7)$$

$$B = b_0 \left( \frac{1 + \sqrt{1 - \xi^2}}{-1 + \sqrt{1 - \xi^2}} \right)^{\beta \left( -1 + \frac{1}{\sqrt{-3-2\omega}} \right)}, \quad (8)$$

$$\phi = \phi_0 \left( \frac{1 + \sqrt{1 - \xi^2}}{-1 + \sqrt{1 - \xi^2}} \right)^{-\frac{\beta}{\sqrt{-3-2\omega}}}, \quad (9)$$

where  $\beta, a_0, b_0, \phi_0$  arbitrary constants. For the prolate case

$$A = a_0 \left( \frac{1 + \xi}{-1 + \xi} \right)^{\beta \left( \frac{1 - \sqrt{-3-2\omega}}{1 + \sqrt{-3-2\omega}} \right)}, \quad (10)$$

$$B = b_0 \left( \frac{1 + \xi}{-1 + \xi} \right)^{\beta}, \quad (11)$$

$$\phi = \phi_0 \left( \frac{1 + \xi}{-1 + \xi} \right)^{\left( \frac{-\beta}{1 + \sqrt{-3-2\omega}} \right)}. \quad (12)$$

These solutions can take some possible forms, depending on the values of arbitrary constants appearing in the solution. Now choose the value  $\omega = -2$  and redefine the arbitrary constant. Then the metric and scalar function become

$$A = 1, \quad (13)$$

$$B = b_0 \left( \frac{1 + \sqrt{1 - \xi^2}}{-1 + \sqrt{1 - \xi^2}} \right)^{\beta}, \quad (14)$$

$$\phi = \phi_0 \left( \frac{1 + \sqrt{1 - \xi^2}}{-1 + \sqrt{1 - \xi^2}} \right)^{-\frac{\beta}{2}}, \quad (15)$$

and

$$A = 1, \quad (16)$$

$$B = b_0 \left( \frac{1 + \xi}{-1 + \xi} \right)^{\beta}, \quad (17)$$

$$\phi = \phi_0 \left( \frac{1 + \xi}{-1 + \xi} \right)^{-\frac{\beta}{2}}. \quad (18)$$

To see that all these metrics is asymptotically flat it is enough to show that the metric components behave in an appropriate way at large  $\xi$ -coordinate values, e.g.,  $g_{\mu\nu} = \eta_{\mu\nu} + O(1/\xi)$  as  $\xi \rightarrow \infty$ . By inspection of the coefficients, we verify that this is so.

It is certainly true that any vacuum solution of Einstein's equations is also a solution of JBD equations (2) with  $\phi$  strictly constant, and that  $\phi = \text{const}$  is the solution of the equation(4) for  $\omega \rightarrow \infty$ . However, this by no means implies that all JBD solutions satisfy Einstein's equations in the limit  $\omega \rightarrow \infty$  or in the limit  $\phi \rightarrow \text{const}$ . In fact, it is easy to show that Einstein's field equations yield only the flat space  $A = 1$ ,  $B = 1$ .

We now deduce the results obtained in the oblate and prolate spheroidal coordinates. Firstly, in both cases, the JBD scalar field appears to play the role of dark matter component. Secondly, the directional components of equation of state of JBD field are anisotropic and in both cases indicate that the JBD scalar field appears to be an "exotic" type of dark matter.

### 3 Discussion

In this article we delineated the qualitative features one would expect from spheroidal scalar field object. It is demonstrated that our model can successfully predict the spheroidal configuration in terms of a self-gravitating spacetime solution to the JBD field equations and reproduce the not spherically-symmetric shape in terms of the non-trivial energy density and anisotropic pressure of the JBD scalar field which was absent in the context of general relativity. We believe that following this hypotheses the shape of galaxy and rotation curve may be explained by action of scalar fields. The solution presented here could be a first approximation at the galactic space-time provided the presence of scalar field. Therefore, it is necessary to study how these results modify the standard method of interpretation rotation data. Further investigation into the nature solutions with view to separating the real rotational effects from the scalar fields anisotropy might be rewarding.

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